Continuous coordinates for all impulsive *pp*-waves

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Abstract

We present a coordinate system for a general impulsive gravitational *pp*-wave in vacuum in which the metric is explicitly continuous, synchronous and "transverse". Also, it is more appropriate for investigation of particle motions.

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The metric of widely known class of vacuum *pp*-waves [1], plane-fronted gravitational waves with parallel rays, can be written in the form

$$\mathrm{d}s^2 = 2\,\mathrm{d}\zeta\mathrm{d}\bar{\zeta} - 2\,\mathrm{d}u\mathrm{d}r - (f+\bar{f})\,\mathrm{d}u^2\ , \tag{1}$$

where $f(u,\zeta)$ is an arbitrary function of the retarded time u and the complex coordinate ζ spanning the plane wave surfaces. Impulsive waves of this type can easily be constructed by considering functions of the form $f = f(\zeta)\delta(u)$ with $\delta(u)$ being the delta function. Although this form is illustrative with the pulse evidently localized along the null hyperplane $u = 0$, such a definition is only formal since the metric components contain the delta function. There are several possibilities for introducing the impulsive *pp*-waves correctly.

Naturally, an impulsive wave can be understood as a limit of suitable sequence of sandwich waves. This was done explicitly for $f = \zeta^2 d(u)$ using various profiles $d(u)$, cf. [2], [3].

Of particular interest is the solution given by Aichelburg and Sexl [4] in which $f = \mu\log\zeta\,\delta(u)$ where μ is a real constant. It was originally obtained by boosting a Schwarzschild black hole to the speed of light while its mass is reduced to zero in an appropriate way. This solution describes an impulsive wave generated by a single null particle at $\zeta = 0$. More general metrics have similarly been obtained from other space-times of the Kerr-Newman family [5]-[8].

A general approach for constructing an arbitrary impulsive *pp*-wave was proposed by Penrose [9]. His "scissors-and-paste" method is based on the removal of the null hyperplane $u = 0$ from Minkowski space-time and re-attaching the halves $\mathcal{M}_-(u<0)$ and $\mathcal{M}_+(u>0)$ by making the identification (a "warp") such that $(\zeta,\bar{\zeta},u=0,r)_{\mathcal{M}_-} \equiv (\zeta,\bar{\zeta},u=0,r+\frac{1}{2}[f(\zeta)+\bar{f}(\bar{\zeta})])_{\mathcal{M}_+}$.

In the presented paper, however, we adopt a different approach for rigorous definition of a general impulsive (vacuum) *pp*-wave by writing it in a simple coordinate system in which the metric components are explicitly *continuous* for all values of u,

$$\mathrm{d}s^2 = 2\left|\mathrm{d}\bar{\eta} - \tfrac{1}{2}u\Theta(u)f''(\eta)\mathrm{d}\eta\right|^2 - 2\,\mathrm{d}u\mathrm{d}v\ , \tag{2}$$

where $\Theta(u)$ is the Heaviside step function ($\Theta = 0$ for $u < 0$, $\Theta = 1$ for $u > 0$) and $f' = \mathrm{d}f/\mathrm{d}\eta$. For the metric (2) the delta function appears only in the components of the curvature tensor. The transformation relating (2) to (1) is

$$\begin{aligned} \zeta &= \eta - \tfrac{1}{2}u\Theta(u)\bar{f}'\ , \\ r &= v - \tfrac{1}{2}\Theta(u)(f+\bar{f}) + \tfrac{1}{4}u\Theta(u)f'\bar{f}'\ , \end{aligned} \tag{3}$$

where $f = f(\eta)$. Note that the transformation is discontinuous at $u = 0$ with the jump in r given by $\frac{1}{2}(f+\bar{f})$ which corresponds to the Penrose identification method.

From (2) it is clear that for linear functions f the metric reduces to Minkowski flat space-time. For other functions f the metrics are non-trivial and describe impulsive gravitational waves. In particular, when $f(\zeta) = C\zeta^n$ where $C = |C| \exp(in\varphi_n)$ is an arbitrary complex constant and n is an integer, it is convenient to introduce two real "polar" coordinates ρ, φ by $\eta = \frac{1}{\sqrt{2}}\rho \exp[i(\varphi - \varphi_n)]$. In these coordinates the metric (2) takes the form

$$\begin{aligned} \mathrm{d}s^2 \;&=\; [1 + u\Theta(u)D\rho^{n-2}]^2(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\varphi^2) \\ &\quad -4u\Theta(u)D\rho^{n-2}[\cos(\tfrac{n}{2}\varphi)\mathrm{d}\rho - \sin(\tfrac{n}{2}\varphi)\rho\mathrm{d}\varphi]^2 - 2\,\mathrm{d}u\mathrm{d}v\ , \end{aligned} \tag{4}$$

where $D = 2^{-n/2}n(n-1)|C|$. We may also assume coordinates $x = \rho\cos\varphi$, $y = \rho\sin\varphi$,

$$\begin{aligned} \mathrm{d}s^2 \;&=\; [1 - 2DA_n u\Theta(u) + D^2(x^2+y^2)^{n-2}u^2\Theta(u)]\,\mathrm{d}x^2 \\ &\quad [1 + 2DA_n u\Theta(u) + D^2(x^2+y^2)^{n-2}u^2\Theta(u)]\,\mathrm{d}y^2 \\ &\quad +4DB_n u\Theta(u)\,\mathrm{d}x\mathrm{d}y - 2\,\mathrm{d}u\mathrm{d}v\ , \end{aligned} \tag{5}$$

where $A_n(x,y) = \mathcal{R}e\,\{(x+iy)^{n-2}\}$, $B_n(x,y) = \mathcal{I}m\,\{(x+iy)^{n-2}\}$.

Again, for $n = 0$ and $n = 1$ the metric (5) reduces to Minkowski space since $D = 0$. For $n = 2$ we get $A_2 = 1$ and $B_2 = 0$ so that (5) gives

$$\mathrm{d}s^2 = (1 - u\Theta(u)|C|)^2\mathrm{d}x^2 + (1 + u\Theta(u)|C|)^2\mathrm{d}y^2 - 2\,\mathrm{d}u\mathrm{d}v\ , \tag{6}$$

which is the well known continuous form of an impulsive plane wave [9]. Solutions (4) with $n > 2$ are unbounded at $\rho = \infty$. Alternatively, they may be written in the form (5) with $A_3 = x$, $B_3 = y$ for $n = 3$, $A_4 = x^2 - y^2$, $B_4 = 2xy$ for $n = 4$ etc.

The metric (4) is very suitable for describing impulsive gravitational *pp*-waves generated by null particles of an arbitrary multipole structure [10] located at $\rho = 0$. These are given by $n = -1, -2, \cdots$. Note that, interestingly, the Aichelburg and Sexl "monopole" solution can also be written in the form (4) with $n = 0$ but $D = -4\mu \neq 0$,

$$\mathrm{d}s^2 = \left[1 + \frac{4\mu}{\rho^2}u\Theta(u)\right]^2\mathrm{d}\rho^2 + \left[1 - \frac{4\mu}{\rho^2}u\Theta(u)\right]^2\rho^2\mathrm{d}\varphi^2 - 2\,\mathrm{d}u\mathrm{d}v\ , \tag{7}$$

which has been introduced in [11], [12].

Considering $u = \frac{1}{\sqrt{2}}(t - z)$ and $v = \frac{1}{\sqrt{2}}(t + z)$ we see that the above coordinate systems are not only continuous but also synchronous (gaussian). Moreover, they are explicitly "transverse", i.e., naturally adapted for describing gravitational *pp*-waves propagating in the z-direction. In particular, they are useful for a study of the motions of free particles (avoiding some of the complications noticed in [13]). Let us assume particles in metric (4) standing at fixed ρ_0, φ_0, z_0 in Minkowski (half) space $u < 0$ ahead of the impulsive wave. They move along geodesics with t measuring their (synchronized) proper times. After the passage of the impulse

they start to move one with respect to the other. In particular, the *relative* distance between two nearby particles having the same φ_0, z_0 but with radial coordinates differing by $\Delta\rho_0$, or having the same ρ_0, z_0 but with a difference $\Delta\varphi_0$, changes for small values of $u = \frac{1}{\sqrt{2}}(t-z_0) > 0$ according to

$$\begin{aligned} \Delta l_\rho &= \Delta\rho_0[1 - \tfrac{D}{\sqrt{2}}\rho_0^{n-2}\cos(n\varphi_0)(t-z_0)] \;, \\ \Delta l_\varphi &= \rho_0\Delta\varphi_0[1 + \tfrac{D}{\sqrt{2}}\rho_0^{n-2}\cos(n\varphi_0)(t-z_0)] \;. \end{aligned} \tag{8}$$

Therefore, when the particles approach in the radial direction they move apart in (perpendicular) tangential direction and vice versa. Locally, the ring of free test particles at any point deforms into an ellipse similarly as is typical for linearized gravitational waves. The effect vanishes (up to the first order in u) in places where $\cos(n\varphi_0) = 0$. For the Aichelburg and Sexl solution ($n = 0, D \neq 0$) the above relative motions do not depend on φ_0. Similarly, the motions are independent of ρ_0 for $n = 2$, i.e., for plane gravitational wave ("homogeneous" *pp*-wave). For $n = 3, 4, \cdots$ the effect given by (8) is more intense with a growing value of ρ_0. On the other hand for solutions with $n = 0, -1, -2, \cdots$ representing impulsive waves generated by null particles of an $n-$pole structure the relative motions of this type vanish as $\rho_0 \to \infty$.

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